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Comments on "Barut-Girardello Coherent States for the Parabolic Cylinder Functions"

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Abstract In Chenaghlou and Faizy (Int. J. Theor. Phys. 2008), the authors claim that they have constructed the Barut-Girardello coherent states for the parabolic cylinder functions. However, we point out here that by introducing these coherent states, Schrödinger was able to put forth the idea of "coherent states of the quantum harmonic oscillator" over eighty years ago. These coherent states are derived not only from the Barut-Girardello eigenvalue equation, but also from the Schrödinger and the Klauder-Perelomov approaches. Thus, contrary to their claim, the authors have not introduced new coherent states. In particular, a wide range of the parabolic cylinder functions do not form an orthonormal basis.

Keywords Coherent states · Special functions · Quantum mechanics

The authors of [1] claim to have constructed the Barut-Girardello coherent states for the parabolic cylinder functions. They write in their conclusion: "Finally in the Hilbert space spanned by the parabolic cylinder eigenstates, the appropriate measure is obtained. It is obvious that other special functions can be studied by the same method used here." This implies that they are responsible for investigating the coherent states for the first time, and that they recommend their method be used to obtain the coherent states of other special functions. Coherent states discussed in [1]—as the first known ones—were discovered by Schrödinger in 1926 [2], and from that time on, have frequently been applied in the literature. Also, it has alternatively been argued that three different approaches—the so called Schrödinger,

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Klauder-Perelomov and Barut-Girardello methods [2–4]—overlap only in the special case of the Heisenberg-Weyl group, that is the dynamical symmetry group of the quantum harmonic oscillator [5, 6]. The only work that has been done in [1] is the change of name from "quantum harmonic oscillator coherent states" to "the Barut-Girardello coherent states for the parabolic cylinder functions", which, of course, does not seem to be appropriate. This is due to the fact that the parabolic cylinder functions in the special cases that their indices take the non negative integers, form an orthonormal basis. The parabolic cylinder functions $D_\nu(x)$ for real given numbers ν are defined in terms of the Whittaker functions as follows [7, 8]

$$D_\nu(x) = 2^{\frac{\nu}{2}+\frac{1}{4}} x^{\frac{-1}{2}} W_{\frac{\nu}{2}+\frac{1}{4},\frac{-1}{4}}\left(\frac{x^2}{2}\right). \tag{1}$$

Generally, when we talk about the parabolic cylinder functions, we refer to the arbitrary real indices ν of $D_\nu(x)$, while for arbitrary ν's, they do not constitute an orthonormal basis. Note that only in the special case that the index ν is considered as a non negative integer, the Whittaker functions are written in terms of the Hermite polynomials as a set of orthonormal functions:

$$D_n(x) = 2^{\frac{-n}{2}} e^{-\frac{x^2}{4}} H_n\left(\frac{x}{\sqrt{2}}\right), \quad n = 0, 1, 2, \ldots. \tag{2}$$

Therefore, the claimed coherent states have actually been written for the functions $D_n(x)$ but not for the functions $D_\nu(x)$. Consequently, the Hilbert space used by the authors of [1] is the same space as the solutions of the quantum harmonic oscillator.

Now, because of necessity, let us remember that the quantum states of the simple harmonic oscillator are described in terms of the Hermite polynomials as follows [7, 8]

$$\Psi_n(x) = \frac{e^{-\frac{x^2}{2}} H_n(x)}{\sqrt{2^n n! \sqrt{\pi}}}, \quad \text{with } H_n(x) = (-1)^n e^{x^2}\left(\frac{d}{dx}\right)^n e^{-x^2}, \tag{3}$$

so that they satisfy the following orthonormality relations

$$\int_{-\infty}^{+\infty} \Psi_n(x)\Psi_m(x)\, dx = \delta_{nm}. \tag{4}$$

It is also well known that the wavefunctions satisfy the eigenvalue equations of the quantum harmonic oscillator as

$$H\Psi_n(x) = \left(n + \frac{1}{2}\right)\Psi_n(x), \quad \text{with } H = \frac{1}{2}\{\hat{a}, \hat{a}^\dagger\} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2, \tag{5}$$

so that, the operators $\hat{a} = \frac{1}{\sqrt{2}}(x + \frac{d}{dx})$ and $\hat{a}^\dagger = \frac{1}{\sqrt{2}}(x - \frac{d}{dx})$ are Hermitian conjugate of each other with respect to the inner product (4). It is again well known that the Heisenberg algebra h_3 [9–11], i.e.

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1, \qquad \left[\hat{a}, 1\right] = \left[\hat{a}^\dagger, 1\right] = 0, \tag{6}$$

are represented by the quantum states of the simple harmonic oscillator as

$$\hat{a}^\dagger \Psi_n(x) = \sqrt{n+1}\Psi_{n+1}(x), \qquad \hat{a}\Psi_n(x) = \sqrt{n}\Psi_{n-1}(x), \qquad 1.\Psi_n(x) = \Psi_n(x). \tag{7}$$

It is needless to say that the above symmetry allows us to construct the coherent states corresponding to the simple harmonic oscillator by any of the Barut-Girardello, Klauder-Perelomov and Schrödinger approaches:

$$|z,x\rangle_{\Psi} = e^{\frac{-|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \Psi_n(x) = \pi^{\frac{-1}{4}} \exp\left(\frac{2zx}{\sqrt{2}} - \frac{x^2}{2} - \frac{z^2}{2} - \frac{|z|^2}{2}\right), \tag{8}$$

where z is a complex variable.

By comparison with [1], it becomes clear that the bases of their Hilbert space, i.e. $\psi_n(x)$'s, involve a change of variable and a scaling of the quantum states of the simple harmonic oscillator as

$$\psi_n(x) := 2^{\frac{-1}{4}} \Psi_n\left(\frac{x}{\sqrt{2}}\right). \tag{9}$$

Then, (4) is converted to an orthonormality relation for new bases

$$\int_{-\infty}^{+\infty} \psi_n(x)\psi_m(x)\, dx = \delta_{nm}. \tag{10}$$

Coherent states corresponding to the bases $\psi_n(x)$ are immediately written as

$$|z,x\rangle_{\psi} = e^{\frac{-|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \psi_n(x) = (2\pi)^{\frac{-1}{4}} \exp\left(zx - \frac{x^2}{4} - \frac{z^2}{2} - \frac{|z|^2}{2}\right) = 2^{\frac{-1}{4}} \left|z, \frac{x}{\sqrt{2}}\right\rangle_{\Psi}. \tag{11}$$

Therefore, the claimed coherent states are the same coherent states of the simple harmonic oscillator, since, both sets of the functions $\psi_n(x)$ and $\Psi_n(x)$ span a Hilbert space.

Let us explain more clearly this delicate problem. In the standard formulation for quantum mechanics, a quantum system is uniquely specified by a Hilbert space and a Hamiltonian operator. This description of quantum mechanics depends on the choice of a fixed weight function for inner product on the Hilbert space. The energy and all other observables are represented by self-adjoint operators so that their measured values are required to be real. This follows from the condition that the total probability should be preserved. The choice of weight function (measure) on the Hilbert space is not unique, since, it is not an observable quantity. Therefore, the most general simultaneous transformation on the measure, the bases of Hilbert space and the operators in a way that the transformed quantities yield an equivalent description of the quantum harmonic oscillator, can be introduced as

$$\Phi_n(x) = \sqrt{\frac{\alpha}{W(x)}} \Psi_n(\alpha x), \tag{12}$$

with the following inner product

$$\int_{-\infty}^{+\infty} \Phi_n(x)\Phi_m(x)W(x)\, dx = \delta_{nm}, \tag{13}$$

in which $W(x)$ and α are arbitrary positive function and arbitrary positive constant, respectively. Coherent states corresponding to the bases $\Phi_n(x)$ are immediately calculated as follows

$$|z,x\rangle_{\Phi} = e^{\frac{-|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \Phi_n(x) = \sqrt{\frac{\alpha}{W(x)}} |z, \alpha x\rangle_{\Psi}. \tag{14}$$

The special case presented in [1], i.e. $|z, x\rangle_{\psi}$, is obtained by considering $W(x) = 1$ and $\alpha = \frac{1}{\sqrt{2}}$. The bases $\Psi_n(x)$ and $\Phi_n(x)$ constitute the same Hilbert spaces with the inner products given by (4) and (13) and with the corresponding coherent states denoted by $|z, x\rangle_{\Psi}$ and $|z, x\rangle_{\Phi}$, respectively. The freedom in the choice of the weight function on the Hilbert space is ignored by the fact that the different weight functions lead to the same Hilbert space structure. Indeed, the unitary representation (7) for the Heisenberg algebra (6) is actually unique up to a scaling of the inner product by an irrelevant positive function $W(x)$, as well as, up to a scaling of the space variable x by an irrelevant positive real number α. Actually, the requirement of self-adjointness of Hamiltonian as well as mutual adjointness of the creation and annihilation operators lead to an essentially fixed choice for the weight function on the Hilbert space. The Hilbert spaces constructed by the bases $\Psi_n(x)$ and $\Phi_n(x)$ are in fact the same and one can not claim that the bases $\Phi_n(x)$ are new representation for the Heisenberg algebra h_3. The latter fact has been reflected in calculating the coherent states as the relation (14), which, in turn shows that the transformation on the space of bases, i.e. (12), is just similar to the transformation on the space of coherent states. Obviously, the number of these transformations is infinite.

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